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Variational principles and fixed point theorems

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A B S T R A C T

The paper concerns fundamental variational principles and the Caristi fixed point theorem. The Brézis–Browder theorem is extended and Altman's theorem is investigated. The notion of istance, an extension of ω-distance, is defined. The new concept enables us to prove some elegant and general variational principles which imply a much stronger form of the Caristi and the Takahashi fixed point theorems. Another consequence is an advanced version of the Ekeland variational principle.

The main results of this paper are theorems of variational type for metric spaces. Their formulation involves a kind of order relation $y \preccurlyeq x$ satisfying the triangle inequality, e.g. defined by $\psi(y) + q(y,x) - \psi(x) \leqslant 0$. Here q, called istance, is a mapping satisfying weaker assumptions than metric (see Definition 15). The idea of istance also includes the notion of ω-distance introduced by Kada, Suzuki and Takahashi [6]. Our main "tool" results (Theorems 21, 22) specify in a precise way the properties of the minimal elements (for $\preccurlyeq$). In consequence one can easily obtain a good-looking extension of the Caristi and Takahashi fixed point theorems (Theorem 24). In addition we get a natural and more general version of Ekeland's variational principle with metric d replaced by d-istance q (Theorem 25). This specialized part of the text is preceded by more general considerations of two kinds:

The initial part of the paper is devoted to the Brézis–Browder theorem, which appears to be a particular case of the results for mappings satisfying the triangle inequality. The proofs are almost immediate.

In the second step (starting from Definition 5) we compare our general Theorem 9 for complete structures (Definition 7) with Altman's well-known result. It seems that these theorems are independent (see Proposition 11 and the subsequent comment). This part ends with Lemma 12 later applied to istances.

First let us present the Brézis–Browder type results which require no topology.

Let X be a nonempty set and $\varphi : X \times X \to R$ a mapping satisfying:

$$\varphi(z,x) \leqslant \varphi(z,y) + \varphi(y,x), \quad x,y,z \in X. \tag{1}$$

Then from $\varphi(x,x) \leqslant 2\varphi(x,x)$ and (1) we obtain

$$0 \leqslant \varphi(x,x) \leqslant \varphi(x,y) + \varphi(y,x), \quad x,y \in X. \tag{2}$$

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The following extension of the Brézis–Browder theorem [2, Theorem 1] was proved as Theorem 13 in [8].

Theorem 1. *Let $\varphi : X \times X \to R$ be a mapping satisfying (1). If for each nonincreasing and bounded sequence $(\varphi(x_n, x_0))_{n\in N}$ there exists an $x \in X$ such that $\liminf_{n\to\infty} \varphi(x, x_n) < 0$, then $\varphi(\cdot, x_0)$ is unbounded below.*

Clearly, the next theorem is stronger than Theorem 1.

Theorem 2. *Let $\varphi : X \times X \to R$ be a mapping satisfying (1). If $\varphi(\cdot, x_0)$ is bounded below, then for each sequence $(\varphi(x_n, x_0))_{n\in N}$ with $\lim_{n\to\infty} \varphi(x_n, x_0) = \gamma = \inf\{\varphi(z, x_0)\colon z \in X\}$ and each $x \in X$ we have $0 \leqslant \liminf_{n\to\infty} \varphi(x, x_n)$.*

Proof. Let $(\varphi(x_n, x_0))_{n\in N}$ be a sequence convergent to γ. Then for each $x \in X$ we have (see (1))

$$0 \leqslant \varphi(x, x_0) - \gamma = \varphi(x, x_0) - \lim_{n\to\infty} \varphi(x_n, x_0) \leqslant \liminf_{n\to\infty} \varphi(x, x_n). \qquad \square$$

Similarly, Lemmas 3 and 4 are equivalent to each other.

Lemma 3. *Let $\varphi : X \times X \to R$ be a mapping satisfying (1). If we have $\lim_{n\to\infty} \varphi(x_n, x_0) = -\infty$ for an $x_0 \in X$, then $\lim_{n\to\infty} \varphi(x, x_n) = \infty$ holds for each $x \in X$.*

Proof. Under our assumptions $\lim_{n\to\infty} \varphi(x, x_n) = \infty$ follows from $\varphi(x, x_0) - \varphi(x_n, x_0) \leqslant \varphi(x, x_n)$ (see (1)). $\square$

Lemma 4. *Let $\varphi : X \times X \to R$ be a mapping satisfying (1). If $\sup\{\varphi(x_0, x_n)\colon n \in N\} < \infty$, for an $x_0 \in X$, then $-\infty < \inf\{\varphi(x_n, x)\colon n \in N\}$ holds for each $x \in X$.*

Now let us present a theorem of Altman's type. Its formulation is preceded by three definitions presented in [8, Definitions 1, 2, 3].

Definition 5. Let X be a nonempty set and $\varphi : X \times X \to R$ a mapping satisfying (1). Then

(i) $\mathcal{S} = (X, \varphi, x_0)$ is a local structure if $\varphi(\cdot, x_0)$ has a finite lower bound,
(ii) $\mathcal{S} = (X, \varphi)$ is a global structure if $\varphi(\cdot, x_0)$ has a finite lower bound for each $x_0 \in X$.

By **structure** we will mean, according to the context, either of the above.

Definition 6. Let $\mathcal{S}$ be a structure. Then $(x_n)_{n\in N}$ is a **Cauchy sequence** in X if for each $\epsilon > 0$ there exists an $n_0 \in N$ such that each $m, n \in N$, $n_0 < m < n$ yield $-\epsilon < \varphi(x_n, x_m) \leqslant 0$.

Definition 7. A structure is **complete** if for each Cauchy sequence $(x_n)_{n\in N}$ in X there exists an $x \in X$ such that $\liminf_{n\to\infty} \varphi(x, x_n) \leqslant 0$.

If $\varphi : X \times X \to R$ satisfies (1), then $\preccurlyeq$ defined by

$$y \preccurlyeq x \quad \text{iff} \quad \varphi(y, x) \leqslant 0, \quad x, y \in X \tag{3}$$

is a transitive relation and therefore any chain in X is contained in a maximal chain [7, Kuratowski lemma, p. 33].

For a mapping $\varphi : X \times X \to R$, an $x_0 \in X$ and $\emptyset \neq A \subset X$ let us adopt

$$\gamma = \inf\bigl\{\varphi(z, x_0)\colon z \in A\bigr\}, \qquad m(A, \varphi) = \bigl\{z \in A\colon \varphi(z, x_0) = \gamma\bigr\}. \tag{4}$$

The essential part of the proof of [8, Theorem 15] yields the following:

Theorem 8. *Let (X, φ, x_0) be a complete structure. Then for $\preccurlyeq$ as in (3) and each maximal chain $A \subset X$ containing a predecessor of x_0, there exists an $x \in m(A, \varphi)$.*

This result can be refined as follows:

Theorem 9. *Let (X, φ, x_0) be a complete structure. Then for $\preccurlyeq$ as in (3) and each maximal chain $A \subset X$ containing a predecessor of x_0, the set $m(A, \varphi)$ (see (4)) is nonempty, and for each $x \in m(A, \varphi)$, $y \in A$ with $\varphi(y, x) \leqslant 0$ (i.e. $y \preccurlyeq x$) the following conditions hold: $\varphi(y, x_0) = \varphi(x, x_0) = \gamma$ (i.e. $y \in m(A, \varphi)$) and $\varphi(y, x) = 0$.*

Proof. In view of Theorem 8 the set $m(A,\varphi)$ is nonempty. Assume $x \in m(A,\varphi)$, $y \in A$ and $\varphi(y,x) \leqslant 0$. Then

$$\gamma \leqslant \varphi(y,x_0) \leqslant \varphi(y,x) + \varphi(x,x_0) \leqslant \gamma$$

means that $y \in m(A,\varphi)$. Furthermore, $\gamma = \varphi(y,x_0)$ and (1) yield

$$0 = \varphi(y,x_0) - \varphi(x,x_0) \leqslant \varphi(y,x),$$

i.e. $\varphi(y,x) = 0$. □

Let us compare Theorem 9 and the following result of Altman [1, Theorem 1.1] (in a formulation adapted to the needs of the present paper):

Theorem 10. *Let $(X,\preccurlyeq)$ be an ordered set and let $\Phi\colon X \times X \to R$ be a mapping such that: for any given y, $\Phi(\cdot, y)$ is bounded from below on $S(y) = \{z \in X\colon z \preccurlyeq y\}$. Assume that Φ and every totally ordered sequence $(x_n)_{n\in N}$ in X satisfy the following conditions:*

(i) *there exists a $y \in X$ such that $y \preccurlyeq x_n$, for all $n \in N$,*
(ii) *$\Phi(x, y) \leqslant 0$ if $x \preccurlyeq y$, for all $x, y \in X$,*
(iii) *$\Phi(x, \cdot)$ is nonincreasing for any given $x \in X$ (i.e. $\Phi(x, y) \leqslant \Phi(x, z)$ if $z \preccurlyeq y$, for all $y, z \in X$),*
(iv) *$\liminf_{n\to\infty} \Phi(x_{n+1}, x_n) = 0$.*

Then for each $x_0 \in X$ there exists a $y \in X$ such that $y \preccurlyeq x_0$ and

$$z \preccurlyeq y \quad \textit{implies} \quad \Phi(z, y) = 0, \quad \textit{for all } z \in X.$$

From Altman's brilliant reasoning it follows that the only requirement for his "order" relation is $y \preccurlyeq y$ (the case of no $z \neq y$ satisfying $z \preccurlyeq y$) and $\Phi(y,y) = 0$. Let us consider a complete structure (X,φ) with X ordered by $\preccurlyeq$ as in (3). Then from $x_{n+1} \preccurlyeq x_n$ and (1) we obtain

$$\varphi(x_{n+1},x_0) - \varphi(x_n,x_0) \leqslant \varphi(x_{n+1},x_n) \leqslant 0, \quad n \in N,$$

i.e. $(\varphi(x_n,x_0))_{n\in N}$ is nonincreasing and bounded, and we get (iv) for φ. In fact $(x_n)_{n\in N}$ is a Cauchy sequence in X (see Definition 6) iff $x_{n+1} \preccurlyeq x_n$, $n \in N$, as then, for all $m < n$ we have

$$\varphi(x_n,x_0) - \varphi(x_m,x_0) \leqslant \varphi(x_n,x_m) \leqslant \varphi(x_n,x_{n-1}) + \cdots + \varphi(x_{m+1},x_m) \leqslant 0.$$

If $(x_n)_{n\in N}$ is a Cauchy sequence in X, then it has a minorant (see Definition 7) and we obtain (i) for φ. Condition (ii) is a direct consequence of the way $\preccurlyeq$ was defined. As regards (iii), we have

$$\varphi(x,y) - \varphi(x,z) \leqslant \varphi(z,y)$$

and $z \preccurlyeq y$ (i.e. $\varphi(z,y) \leqslant 0$) implies $\varphi(x,y) \leqslant \varphi(x,z)$, which means that $\varphi(x,\cdot)$ is nonincreasing for each $x \in X$ (in a similar way one can prove that $\varphi(\cdot,y)$ is nondecreasing).

Our reasoning yields:

Proposition 11. *If (X,φ) is a complete structure, then for $\preccurlyeq$ as in (3) and any Cauchy sequence $(x_n)_{n\in N}$ in X the assumptions* (i), …, (iv) *of [1, Theorem 1.1] are satisfied with Φ replaced by φ.*

Nevertheless, we do not assume $\varphi(y,y) = 0$ and Altman says nothing about $\Phi(y,x_0)$. It seems that Theorem 9 and Altman's theorem are independent, while the assumptions of Theorem 9 are more consistent.

Let us prove another property of complete structures. This result will be used in some further proofs.

Lemma 12. *Let (X,φ,x_0) be a complete structure and $A \subset X$ a maximal chain containing a predecessor of x_0. Assume that for an $x \in A$ and a nonempty set $V = \{y \in A\colon \varphi(y,v) \leqslant 0\}$*

$$\varphi(y,x) > 0, \quad \textit{for each } y \in V \setminus \{x\} \tag{5}$$

holds. Then $x \in m(A,\varphi)$ (see (4)) is the unique smallest element of A and $0 < \varphi(y,x)$ is satisfied for each $y \in X \setminus \{x\}$ (i.e. x is minimal in X).

Proof. Suppose $x \notin m(A,\varphi)$, i.e. there exists a $y \in V \cap m(A,\varphi) \neq \emptyset$ (see Theorem 9) such that $\varphi(y,x_0) < \varphi(x,x_0)$. Then from

$$0 < \varphi(x,x_0) - \varphi(y,x_0) \leqslant \varphi(x,y)$$

it follows that $\varphi(y,x) \leqslant 0$ (x, y belong to a chain) and $y = x$ (see (5)) – a contradiction. Therefore $x \in m(A,\varphi)$. From (5) it follows that no other element of A precedes x. In addition x is minimal as A is maximal. □

Structure is a general concept. In the sequel we present a special type of it (see Lemma 18).

Let us recall the notion of ω-distance introduced in [6].

Definition 13. Let (X,d) be a metric space. Then a mapping $p\colon X \times X \to [0,\infty)$ is an ***ω*-distance** on X if the following conditions are satisfied:

(i) $p(x,z) \leqslant p(x,y) + p(y,z)$, $x, y, z \in X$,
(ii) $p(x,\cdot)$ is lower semicontinuous, $x \in X$,
(iii) for each $\epsilon > 0$ there exists a $\delta > 0$ such that $p(z,x) \leqslant \delta$ and $p(z,y) \leqslant \delta$ imply $d(x,y) \leqslant \epsilon$.

The authors present numerous examples and properties of ω-distances. Let us present another idea.

Definition 14. Let $q\colon X \times X \to [0,\infty)$ be a mapping. Then $(x_n)_{n\in N}$ is a **Cauchy sequence** in (X,q) if for each $\epsilon > 0$ there exists an $n_0 \in N$ such that each $m,n \in N$, $n_0 < m < n$ yield $q(x_n,x_m) < \epsilon$.

Definition 15. Let (X,d) be a metric space. A mapping $q\colon X \times X \to [0,\infty)$ is a ***d*-istance** in X if the following conditions are satisfied:

$$q(z,x) \leqslant q(z,y) + q(y,x), \quad x, y, z \in X, \tag{6}$$

$$q(\cdot,x) \text{ is lower semicontinuous}, \quad x \in X, \tag{7}$$

$$\text{each Cauchy sequence in } (X,q) \text{ is a Cauchy sequence in } (X,d). \tag{8}$$

The properties of ω-distances include the following one (see [6, Lemma 1(iii)]):

Lemma 16. *Let (X,d) be a metric space and let p be an ω-distance in X. If for a sequence $(\alpha_n)_{n\in N}$ in R convergent to zero*

$$p(x_m,x_n) \leqslant \alpha_m, \quad m < n,\ m,n \in N$$

holds, then $(x_n)_{n\in N}$ is a Cauchy sequence in (X,d).

Lemma 16 yields the following:

Corollary 17. *Let (X,d) be a metric space and p an ω-distance on X. Then q defined by $q(y,x) = p(x,y)$, $x, y \in X$ is a d-istance in X.*

Istances have nice properties.

Lemma 18. *Let (X,d) be a complete metric space, q a d-istance in X and $\psi : X \to R$ a lower semicontinuous mapping bounded below. Then for $\varphi : X \times X \to R$ defined by*

$$\varphi(y,x) = \psi(y) + q(y,x) - \psi(x), \quad x, y \in X, \tag{9}$$

(X,φ) is a complete structure.

Proof. Let $x_0 \in X$ be arbitrary. In view of (6) φ satisfies (1) and clearly, $\varphi(\cdot,x_0)$ is bounded below. The inequality

$$-\epsilon < \varphi(x_n,x_m) = \psi(x_n) + q(x_n,x_m) - \psi(x_m) \leqslant 0$$

(see Definition 6) means

$$\psi(x_m) - \psi(x_n) - \epsilon < q(x_n,x_m) \leqslant \psi(x_m) - \psi(x_n)$$

and therefore $(\psi(x_n))_{n\in N}$ is nonincreasing ($m < n$) and convergent, as ψ is bounded below. In view of (8) $(x_n)_{n\in N}$ converges, say to a point x ((X,d) is complete). From the lower semicontinuity of ψ, $q(\cdot,x_m)$ it follows that

$$\varphi(x,x_m) = \psi(x) + q(x,x_m) - \psi(x_m) \leqslant 0$$

and

$$\liminf_{n\to\infty} \varphi(x,x_n) = \liminf_{n\to\infty}\big(\psi(x) + q(x,x_n) - \psi(x_n)\big) \leqslant 0.$$

Thus (X,φ,x_0) is a complete structure for each $x_0 \in X$. □

Proposition 19. *Let (X,d) be a metric space. Then any d-istance q in X satisfies*

$$q(x,y)=0 \quad \text{and} \quad q(y,x)=0 \quad \text{imply} \quad x=y, \quad x,y\in X. \tag{10}$$

Proof. Assume $q(x,y)=0$ and $q(y,x)=0$. Let us adopt $x_{2k-1}=x$ and $x_{2k}=y$, $k\in N$. Clearly, $q(x_n,x_m)=0$ holds for each $m\neq n$, $m,n\in N$, and in view of (8) $(x_n)_{n\in N}$ is a Cauchy sequence in (X,d), which means $x=y$. □

The next property is more advanced.

Lemma 20. *Let (X,d) be a complete metric space and q a d-istance in X. Assume that $C\subset X$ is a nonempty maximal chain for*

$$x\leqslant y \quad \text{iff} \quad q(x,y)=0. \tag{11}$$

Then there exists an $x\in C$ satisfying

$$q(x,y)=0, \quad y\in C\setminus\{x\}, \quad \text{and} \quad 0<q(y,x), \quad y\in X\setminus\{x\}. \tag{12}$$

Proof. Let $x_1\in C_1=C$ be arbitrary, $\rho_1=1$ and $C_2=\{x\in C_1\colon\ q(x,x_1)=0\}$. Let us adopt $\rho_2=\min\{1,\sup\{d(x,x_1)\colon\ x\in C_2\}\}$. Now we choose $x_2\in C_2$ such that $\rho_2/2\leqslant d(x_2,x_1)$. If $C_n\neq\emptyset$, ρ_n and x_n are defined, then we adopt $C_{n+1}=\{x\in C_n\colon\ q(x,x_n)=0\}$. If $C_{n+1}\neq\emptyset$ then $\rho_{n+1}=\min\{1,\sup\{d(x,x_n)\colon\ x\in C_{n+1}\}\}$ and we choose $x_{n+1}\in C_{n+1}$ such that $\rho_{n+1}/2\leqslant d(x_{n+1},x_n)$. Assume that the sequence of points x_n is infinite. Then we have

$$q(x_n,x_m)\leqslant q(x_n,x_{n-1})+\cdots+q(x_{m+1},x_m)=0, \quad m<n$$

and therefore (see (8)) $(x_n)_{n\in N}$ is a Cauchy sequence in (X,d). Consequently $\{C_n\colon\ n\in N\}$ is a decreasing family of sets with $\lim_{n\to\infty} dia(C_n)=0$. On the other hand, for $x=\lim_{n\to\infty}x_n$ ((X,d) is complete) we have

$$0\leqslant q(x,x_m)\leqslant q(x_n,x_m)=0, \quad m<n,$$

as $q(\cdot,x_m)$ is lower semicontinuous and consequently, $x\in C_n$, $n\in N$. This means that $\bigcap\{C_n\colon\ n\in N\}=\{x\}$. If the sequence of points x_n is finite, then we denote its last element as x. For any $y\in C\setminus\{x\}$ there exists an n such that $y\notin C_n$, i.e. $q(x_n,y)=0$. For $x\neq x_n$ we obtain

$$q(x,y)\leqslant q(x,x_n)+q(x_n,y)=0$$

which means that $x\in C$. If $q(y,x)=0$ then $y\in C$, and the second part of condition (12) follows from (10). □

For a mapping $\psi: X\to R$ let us adopt

$$\beta=\inf\bigl\{\psi(z)\colon\ z\in X\bigr\}, \qquad B=\bigl\{z\in X\colon\ \psi(z)=\beta\bigr\}. \tag{13}$$

Now we are ready to prove the following far extension of [6, Theorem 1]:

Theorem 21. *Let (X,d) be a complete metric space, q a d-istance in X, and $\psi: X\to R$ a lower semicontinuous mapping bounded below. Let us adopt $y\preccurlyeq x$ iff $\psi(y)+q(y,x)-\psi(x)\leqslant 0$, $x,y\in X$, and assume that the following holds (see (13)):*

$$\text{for each } x\in X\setminus B \text{ there exists a } y\in X\setminus\{x\} \text{ such that } y\preccurlyeq x. \tag{14}$$

Then for any $x_0\in X\setminus B$, each maximal chain $A\subset X$ containing x_0 has a unique smallest element x, in addition satisfying:

(i) $\psi(x)=\inf\{\psi(z)\colon\ z\in X\}$ (*i.e.* $x\in B$),
(ii) $\psi(x)+q(x,x_0)-\psi(x_0)=\inf\{\psi(z)+q(z,x_0)-\psi(x_0)\colon\ z\in A\}\leqslant 0$,
(iii) $0<\psi(y)+q(y,x)-\psi(x)$, *for each* $y\in X\setminus\{x\}$.

Proof. The relation $\preccurlyeq$ is transitive and in view of Kuratowski's lemma [7, p. 33] there exists a maximal chain A containing any $x_0\in X\setminus B$ (see (14)). Let us adopt $\alpha=\inf\{\psi(z)\colon\ z\in A\}$ and suppose $\alpha<\psi(x)$, for each $x\in A$. Then there exists a sequence $(x_n)_{n\in N}$ in A such that $(\psi(x_n))_{n\in N}$ decreases to α. Condition $x_m\preccurlyeq x_n$ for $m<n$ means that

$$0\leqslant q(x_m,x_n)\leqslant\psi(x_n)-\psi(x_m)<0,$$

which is impossible. Now from $x_n\preccurlyeq x_m$ ($x_n,x_m\in A$) we obtain

$$q(x_n,x_m)\leqslant\psi(x_m)-\psi(x_n), \quad m,n\in N, \ m<n$$

and $(x_n)_{n\in N}$ is a Cauchy sequence in (X,d) (see (8)), as ψ is bounded below. Let us adopt $v=\lim_{n\to\infty} x_n$. From the lower semicontinuity of ψ, $q(\cdot,x_m)$ it follows that $\psi(v)\leqslant\alpha$ and

$$\psi(v)+q(v,x_m)-\psi(x_m)\leqslant 0, \quad m\in N.$$

On the other hand, for any $y\in A$ there exists an x_n such that $\psi(x_n)<\psi(y)$ and consequently ($x_n, y\in A$)

$$\psi(x_n)+q(x_n,y)-\psi(y)\leqslant 0$$

must hold, as

$$0\leqslant q(y,x_n)\leqslant\psi(x_n)-\psi(y)<0$$

is false. Now we obtain (see (6))

$$\psi(v)+q(v,y)-\psi(y)\leqslant\psi(v)+q(v,x_n)-\psi(x_n)+\psi(x_n)+q(x_n,y)-\psi(y)\leqslant 0.$$

This dependency means that there exists a $v\in A$ such that $\psi(v)=\alpha$. If v is not a (unique) smallest element of A, then from $y\preccurlyeq v$ for a $y\in A\setminus\{v\}$ it follows that

$$0\leqslant q(y,v)\leqslant\psi(v)-\psi(y),$$

and we obtain $\psi(y)=\alpha$, $q(y,v)=0$. Now it is seen that the set $V=\{y\in A\colon\ y\preccurlyeq v\}$ is nonempty and $V\subset C$ (see (11)). On the other hand, $Y=\{z\in X\colon\ \psi(z)+q(z,v)-\psi(v)\leqslant 0 \text{ and } \psi(z)\leqslant\psi(v)\}$ is complete, nonempty and $V\subset Y$. Now we apply Lemma 20 to Y, $C\cap Y$ in place of X, C, respectively. For x as in (12) and any $y\in V\setminus\{x\}$ we have

$$\varphi(x,y)=\psi(x)+q(x,y)-\psi(y)=\psi(x)-\psi(y)\leqslant 0,$$

i.e. $x\in m(A,\varphi)$ (see (4)) and x is the unique smallest element of $A\cap Y$ (see Lemmas 18, 12) and of A, as $V\subset Y$. Conditions (ii), (iii) follow. In view of (14), $x\in B$ (otherwise x would have a predecessor), i.e. (i) is satisfied. $\square$

The proof of Theorem 21 would be much easier for q satisfying: $q(y,x)=0$ implies $x=y$, $x,y\in X$ (Lemma 20 could be disregarded). Is such a condition satisfied for all istances or even ω-distances ([6] does not contain a counterexample)?

A reasoning similar to the one presented in the above proof yields:

Theorem 22. *Let (X,d) be a complete metric space, q a d-istance in X, and $\psi: X\to R$ a lower semicontinuous mapping bounded below. Let us adopt $y\preccurlyeq x$ iff $\psi(y)+q(y,x)-\psi(x)\leqslant 0$, $x,y\in X$, and assume that the following condition holds:*

$$\textit{for each } x\in X \textit{ there exists a } y \textit{ such that } y\preccurlyeq x. \tag{15}$$

Then for any $x_0\in X$, each maximal chain $A\subset X$ containing x_0 has a unique smallest element x, in addition satisfying:

(i) $\psi(x)=\inf\{\psi(z)\colon\ z\in A\}$,
(ii) $\psi(x)+q(x,x_0)-\psi(x_0)=\inf\{\psi(z)+q(z,x_0)-\psi(x_0)\colon\ z\in A\}\leqslant 0$,
(iii) $0<\psi(y)+q(y,x)-\psi(x)$, *for each* $y\in X\setminus\{x\}$,
(iv) $q(x,x)=0$.

Proof. In view of (15) there exists a maximal chain A containing x_0. Now the remaining part of the proof of Theorem 21 works. From (15) and the fact that x is the smallest element of A it follows that

$$0\leqslant q(x,x)=\psi(x)+q(x,x)-\psi(x)=\varphi(x,x)\leqslant 0.$$

The last sentence of the proof of Theorem 21 cannot be added and we do not know if $x\in B$. $\square$

Let 2^Y be the family of all subsets of Y. We say that $F: X\to 2^Y$ is a (multivalued) mapping if $F(x)\neq\emptyset$, for all $x\in X\neq\emptyset$.

A consequence of Theorem 22 is the following:

Theorem 23. *Let (X,d) be a complete metric space, q a d-istance in X, and $\psi: X\to R$ a lower semicontinuous mapping bounded below. Assume $X\subset Y$ and $F: X\to 2^Y$ is a mapping satisfying:*

$$\textit{for each } x\in X\setminus F(x) \textit{ there exists a } y\in X\setminus\{x\} \textit{ such that } \psi(y)+q(y,x)-\psi(x)\leqslant 0. \tag{16}$$

Then F has a fixed point.

Proof. Suppose F has no fixed point. Then Theorem 22 applies and (iii) contradicts (16). $\square$

The subsequent theorem extends the theorems of Caristi [3, Theorem (2.1)′], Takahashi [5, Theorem 5] and Theorem 2 from [6].

Theorem 24. *Let (X, d) be a complete metric space, q a d-istance in X, and $\psi : X \to R$ a lower semicontinuous mapping bounded below. Let us adopt $y \preccurlyeq x$ iff $\psi(y) + q(y, x) - \psi(x) \leqslant 0$, $x, y \in X$, and assume that $X \subset Y$ and $F : X \to 2^Y$ is a mapping satisfying:*

$$\text{for each } x \in X \text{ there exists a } y \in F(x) \text{ such that } y \preccurlyeq x. \tag{17}$$

Then for any $x_0 \in X$, each maximal chain $A \subset X$ containing x_0 has a unique smallest element x, in addition satisfying conditions (i), …, (iv) *of Theorem* 22 *and such that $x \in F(x)$.*

Proof. In view of (17) Theorem 22 works. Now condition (17) and (iii) mean $x \in F(x)$. $\square$

The subsequent theorem extends Ekeland's variational principle [4, Theorem 1] and [6, Theorem 3] as q need not be a metric.

Theorem 25. *Let (X, d) be a complete metric space, q a d-istance in X, and $\psi : X \to R$ a lower semicontinuous mapping bounded below. Then the following are satisfied:*

(i) *for each $x_0 \in X$ there exists an $x \in X$ such that $\psi(x) \leqslant \psi(x_0)$ and $\psi(x) - q(y, x) < \psi(y)$, for each $y \in X \setminus \{x\}$,*
(ii) *for any $\epsilon > 0$ and each $x_0 \in X$ with $q(x_0, x_0) = 0$ and $\psi(x_0) \leqslant \epsilon + \inf\{\psi(z){:}\ z \in X\}$ there exists an $x \in X$ such that $\psi(x) \leqslant \psi(x_0)$, $q(x, x_0) \leqslant 1$ and $\psi(x) - \epsilon q(y, x) < \psi(y)$, for each $y \in X \setminus \{x\}$.*

Proof. The reasoning based on our Theorem 22 is similar to the one presented in the proof of [6, Theorem 3]. The set $Y = \{z \in X{:}\ \psi(z) \leqslant \psi(x_0)\}$ is complete. Suppose that for each $x \in Y$ there exists a $y \in Y \setminus \{x\}$ such that $\psi(y) + q(y, x) - \psi(x) \leqslant 0$. Then by Theorem 22(iii) there exists an $x \in Y$ such that $0 < \psi(y) + q(y, x) - \psi(x)$ – a contradiction, i.e. (i) is proved for Y in place of X. Condition (i) is trivial for $y \in X \setminus Y$.

Now let us consider ϵq in place of q. The set $Z = \{z \in X{:}\ \psi(z) + \epsilon q(z, x_0) - \psi(x_0) \leqslant 0\}$ is complete and $x_0 \in Z$. In view of (i) there exists an $x \in Z$ such that $0 < \psi(y) + \epsilon q(y, x) - \psi(x)$, for each $y \in Z \setminus \{x\}$. On the other hand, from (6) it follows that any y not satisfying the last inequality must belong to Z. Consequently, our inequality is valid for each $y \in X \setminus \{x\}$. From $\epsilon q(x, x_0) \leqslant \psi(x_0) - \psi(x)$ $(x \in Z)$ and the second assumption of (ii) we obtain

$$q(x, x_0) \leqslant \left[\psi(x_0) - \psi(x)\right]/\epsilon \leqslant \left[\psi(x_0) - \inf\left\{\psi(z){:}\ z \in X\right\}\right]/\epsilon \leqslant 1. \qquad \square$$

By using exactly the same technique as in [6] for $p(x, y) = q(y, x)$, we obtain the following extension (see Corollary 17) of [6, Theorem 4]:

Theorem 26. *Let (X, d) be a complete metric space, q a d-istance in X, and let $f{:}\ X \to X$ be a mapping. Assume that for an $\alpha < 1$ the following hold:*

(i) $q(f^2(x), f(x)) \leqslant \alpha q(f(x), x)$, $x \in X$,
(ii) $0 < \inf\{q(y, x) + q(f(x), x){:}\ x \in X\}$, $y \in X \setminus \{f(y)\}$.

Then f has a fixed point, and $x = f(x)$ implies $q(x, x) = 0$.

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